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Use of boundary conditions of the third kind to model heat conduction between two proximate rough surfaces separated by an insulator

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Abstract—Steady-state heat conduction through an insulating layer separating the surface of a rough, isothermal body (e.g. a sphere) from an isothermal semi-infinite region bounded by a rough plane is modeled by employing Robin boundary conditions with a 'slip' coefficient on the smoothed body surface and plane. This model is proposed for circumstances where the roughness asparities are comparable in size to the nominal gap width (and each asparity is much smaller than the characteristic body radius). It is shown that such slip, however small, serves to remove the logarithmic gap-width singularity in the contact case. An attempt is made to rationalize use of this 'effective' boundary condition of the third kind, as well as to obtain an estimate for the slip coefficient appearing therein in terms of the scale of the asparities relative to the gap width. In this first attempt to develop a theory, particular attention is paid to the elementary case where the body is spherical in shape. Copyright © 1996 Elsevier Science Ltd.

1. INTRODUCTION

It is known theoretically [1–4] that the rate of heat transfer between two closely proximate, highly conducting spheres (or, equivalently, between a sphere and a plane) separated by a poor conductor or insulator becomes logarithmically infinite as the gap width ε (made nondimensional with the sphere radius a) tends to zero. As a result, the rate of heat transfer depends critically upon the relative closeness of the two surfaces. The contact case, where the adjacent surfaces touch, is physically indeterminate since issues of the deformation of the surfaces pertaining to the Hertzian contact come to the fore. However, at gap widths existing well before this contact issue arises, one must confront the fact that all real surfaces are rough rather than smooth. As in the corresponding electrical situation, this issue particularly comes to the fore when the gap width becomes comparable to the scale of the asparities.

While heat conduction problems involving roughness in contact problems have received some attention in the literature [5–7], the rational resolution of the issues raised by the phenomenon is far from clear. The net effect of this is to leave unanswered the basic question of, what is the magnitude of the conduction heat-transfer rate? This rate, on the basis of casual inspection, appears to be both non-zero and unique for a given pair of (rough) surfaces, prescribed temperatures, and a given insulator—independently of precisely how the touching asparities of the juxtaposed surfaces are configured relative to one another.

In this paper a technological escape from the point contact singularity dilemma is proposed. The proposition consists of replacing the true thermal boundary conditions on the rough, proximate surfaces by an apparent thermal 'slip' boundary condition (i.e. by a boundary condition of the third kind) on the hypothetical *smooth* surfaces. In this context, it is shown that the inclusion of a slip coefficient, however small, removes the contact singularity in the sphere–plane conduction problem. While replacement of the true boundary condition by an apparently artificial one may appear to be *ad hoc*, the slip proposal is rendered sensible at the conclusion of the paper by giving a plausible, though somewhat crude, derivation of this effective boundary condition. Concomitantly, the spatially periodic geometrical model used to advance this slip argument for rough surfaces simultaneously provides an order-of-magnitude estimate of the slip coefficient in terms of the scale of the roughness elements and the gap width.

A preview of the contents of the present paper is as follows. Section 2 considers the problem of a smooth

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NOMENCLATURE

a	sphere radius
$A(v)$	function in equation (5)
b	center-to-center spacing between bumps
B	misalignment distance of opposing protrusions in Fig. 2
$B(v)$	function in equation (5)
c	radius of bump circle in Fig. 1
d	distance between planes in Fig. 1
D	distance between walls in Fig. 2
$f(y)$	integral equation function in equation (22)
f_n	Chebyshev polynomial expansion coefficient in equation (27)
$\langle f \rangle$	mean value of f over the range $0 \leqslant B \leqslant L$
g_{nm}	expansion coefficient in equation (28)
G	Green's function in equation (21)
$G^\bullet$	regular function in equation (25)
H	Heaviside function
J_0	Bessel function of order zero
k	thermal conductivity of insular layer
L	half the separation distance between adjacent protrusions in Fig. 2
m	summation index
n	normal distance or summation index
N	finite truncation integer index
O	gauge or order symbol
p	probability density
Q	total heat flux or mean heat flux density
r	circular cylindrical coordinate
T	temperature
$T_{2n-1}(y)$	Chebyshev polynomial
v	dummy integration variable in equation (5)
W	dimensionless limiting value of mean heat flux density
$W^\bullet$	dimensionless form of integral (15) at $\varepsilon = 0$
x	Cartesian coordinate
y	Cartesian coordinate
Y_0, Y_1	location of fictitious planar boundaries in equation (30)
z	Cartesian or circular cylindrical coordinate.

Greek symbols

δ	dimensionless thermal slip or impedance coefficient
δ_{nm}	Kronecker delta
ε	nondimensional gap width
η	tangent–sphere coordinate in equation (4)
θ	dimensionless temperature in equation (17)
λ	misalignment distance in Fig. 1
λ_0	cutoff distance in equation (15)
ξ	tangent–sphere coordinate in equation (4)
Σ'	special summation symbol defined following equation (28)
ϕ	circular cylindrical coordinate
χ	dummy angular integration variable.

Subscripts

0, 1	designates either of two adjacent surfaces.

Supercript

$'$	denotes derivative.

Overline symbol

$\wedge$	relates to use of fictitious boundary conditions on smoothed surfaces.

rigid sphere touching a smooth rigid plane (the latter bounding a semi-infinite domain) with linear slip boundary conditions of the third kind imposed on each surface. The respective thermal conductivities of each of these two bodies is regarded as sufficiently large to make each body effectively isothermal (corresponding to temperatures T_0 and T_1, respectively). With $\delta_0 \ll 1$ and $\delta_1 \ll 1$ defined as small slip coefficients (rendered nondimensional with the sphere radius) on the sphere and plane, respectively, the nondimensional heat transfer rate across the insulating gap is shown to be functionally dependent upon $\ln(\delta_0+\delta_1)^{-1}$. Hence, provided that either δ_0 or δ_1 is non-zero, the heat transfer rate for the contact problem is nonsingular.

Section 3 puts the classical no-slip, small gap (smooth), sphere–plane wall case into perspective, beginning with the observation that the comparable nondimensional heat transfer rate for this case is functionally dependent upon $\ln \varepsilon^{-1} (\varepsilon \ll 1)$, and hence is singular in the touching case, $\varepsilon \to 0$. In this context, the three distinct limiting cases, namely: (i) $\varepsilon \gg \delta_0+\delta_1$, (ii) $\varepsilon = O(\delta_0+\delta_1)$ and (iii) $\varepsilon \ll \delta_0+\delta_1$ are discussed.

Section 4 uses a spatially periodic roughness model composed of bumps (identical spherical caps) on two juxtaposed planes to lend credibility to the slip model for the effective, smoothed surfaces. In particular, it is shown that whereas logarithmic singularity is obtained when the oppositely situated asparities approach contact, the *net* rate of heat transfer between the semi-infinite planar domains is, nevertheless, nonsingular. This order-of-magnitude calculation is used to obtain an estimate for the slip coefficient δ (assumed to be the same on both surfaces).

Whereas the analysis of Section 4 leads only to an inequality rather than a definite estimate for the slip coefficient, Section 5 quantifies and otherwise makes credible the existence of an effective slip coefficient, at least for a particular case, namely 2-D heat flow between parallel, conducting planes disturbed by periodic arrays of thin conducting protrusions perpendicular to the walls. Direct numerical calculations for this case reveal the robustness of the concept of modeling such 'rough' planar surfaces via a thermal 'slip' coefficient.

2. FICTITIOUS BOUNDARY CONDITIONS ON SPHERE AND PLANE

Suppose that a perfectly conducting sphere of radius a is held at temperature $T = T_0$ and constrained to remain in contact with a perfectly conducting half space whose temperature is $T = T_1$. Between the plane wall and the sphere surface lies a conducting medium with thermal conductivity k; the temperature difference $T_1 \neq T_0$ is facilitated by imposing fictitious boundary conditions on the temperature field T in the conducting region. Choose a Cartesian coordinate system that is fixed in the sphere, with the origin situated at the point of contact, the plane wall at $z = 0$, and the center of the sphere at $(0, 0, a)$.

For steady-state heat conduction, the temperature $T(x, y, z)$ in the conducting domain $z > 0$ external to the sphere $x^2+y^2+(z-a)^2 = a^2$ satisfies Laplace's equation,

$$\nabla^2 T = 0 \tag{1}$$

subject to the boundary conditions

$$T - a\delta_0 \frac{\partial T}{\partial n} = T_0 \quad \text{at } x^2+y^2+z^2 = 2az \tag{2}$$

$$T - a\delta_1 \frac{\partial T}{\partial z} = T_1 \quad \text{at } z = 0 \tag{3}$$

where $\partial/\partial n$ denotes differentiation along a normal directed out of the sphere; here, δ_0, δ_1 are small, positive, nondimensional constants that are not necessarily of the same order and not both zero.

With (r, ϕ, z) circular cylindrical coordinates possessing the same origin as the preceding Cartesian system, the geometric configuration of the problem suggests introducing tangent-sphere coordinates (ξ, ϕ, η) defined by

$$(z, r) = \frac{a}{(\xi^2+\eta^2)}(\xi, \eta) \tag{4}$$

whence the conducting region corresponds to the domain $0 \leqslant \xi \leqslant \frac{1}{2}$, $0 \leqslant \eta < \infty$, $-\pi < \phi \leqslant \pi$. Then, as given by Morse and Feshbach [8], the solution of equation (1) may be expressed as

$$T = T_0 + (T_1 - T_0)(\xi^2+\eta^2)^{1/2} \int_0^\infty [A(v)\cosh v\xi + B(v)\sinh v\xi] J_0(v\eta)\,\mathrm{d}v. \tag{5}$$

The boundary condition (3) at the plane $\xi = 0$ requires that

$$1 = \eta \int_0^\infty [A(v) - \delta_1 \eta^2 v B(v)] J_0(v\eta)\,\mathrm{d}v$$

$$= \eta \int_0^\infty \left[A + \delta_1 \frac{\mathrm{d}}{\mathrm{d}v}\left(v\frac{\mathrm{d}B}{\mathrm{d}v}\right)\right] J_0(v\eta)\,\mathrm{d}v$$

after two integrations by parts involving suitable assumptions on the behavior of $B(v)$ as $v \to 0$. Hence,

$$A + \delta_1 \frac{\mathrm{d}}{\mathrm{d}v}\left(v\frac{\mathrm{d}B}{\mathrm{d}v}\right) = 1. \tag{6}$$

Meanwhile, since the definition (4) shows that

$$a\frac{\partial T}{\partial n} = -\left(\frac{1}{4}+\eta^2\right)\left(\frac{\partial T}{\partial \xi}\right)_{\xi=1/2}$$

the boundary condition (2) on the sphere yields, after a similar calculation,

$$A\cosh\frac{1}{2}v + B\sinh\frac{1}{2}v - \delta_0\left[\left(v\frac{\mathrm{d}^2B}{\mathrm{d}v^2} + v\frac{\mathrm{d}A}{\mathrm{d}v} + \frac{\mathrm{d}B}{\mathrm{d}v}\right) \times \cosh\frac{1}{2}v + \left(v\frac{\mathrm{d}^2A}{\mathrm{d}v^2} + v\frac{\mathrm{d}B}{\mathrm{d}v} + \frac{\mathrm{d}A}{\mathrm{d}v}\right)\sinh\frac{1}{2}v\right] = 0. \tag{7}$$

Substitution of equation (6) into equation (7) then furnishes the following differential equation for $B(v)$:

$$B - (\delta_0+\delta_1)\left(v\frac{\mathrm{d}^2B}{\mathrm{d}v^2} + \frac{\mathrm{d}B}{\mathrm{d}v}\right)\coth\frac{1}{2}v - \delta_0 v\frac{\mathrm{d}B}{\mathrm{d}v} + O(\delta_0\delta_1) = -\coth\tfrac{1}{2}v. \tag{8}$$

The total heat flux Q from the plane to the sphere is given by

$$Q = 2\pi k \int_0^\infty r\,\mathrm{d}r\left(-\frac{\partial T}{\partial z}\right)_{z=0}$$

$$= -2\pi(T_1 - T_0)ka \int_0^\infty B(v)\,\mathrm{d}v \tag{9}$$

in which the integral converges under the assumed behavior of $B(v)$ as $v \to 0$. The known singularity of Q in the dual limit δ_0 and $\delta_1 \to 0$ is easily demonstrated by observing that in this limit, equation (8) yields

$$B = -\coth\frac{1}{2}v \sim -\frac{2}{v} \quad \text{as } v \to 0.$$

However, if $\delta_0+\delta_1 \neq 0$, equation (8) shows that

$$B_1 \sim v/(\delta_0+\delta_1) + O(v^3) \quad \text{as } v \to 0.$$

Hence, application of at least one of the pair of fictitious boundary conditions (i.e. $\delta_0 \neq 0$ or $\delta_1 \neq 0$) removes the singularity, yielding a convergent integral in the expression (9) for the heat flux.

The asymptotic behavior of Q for $\delta_0+\delta_1 \ll 1$ may be determined from the singular solution in the limit $\delta_0+\delta_1 = 0$. When $B = -\coth\frac{1}{2}v$ is substituted into equation (8), the terms multiplying $(\delta_0+\delta_1)$ become comparable with the others when $v = O[(\delta_0+\delta_1)^{1/2}]$. Thus, from equation (9),

$$Q \sim 4\pi(T_1-T_0)ka\int_{(\delta_0+\delta_1)^{1/2}}^{1} v^{-1}\,dv$$

$$= 2\pi(T_1-T_0)ka\ln[(\delta_0+\delta_1)^{-1}]. \quad (10)$$

Note the symmetric dependence on the parameters δ_0 and δ_1, either one of which (but not both) can be allowed to be zero in the above argument.

3. APPROACH TO THE ABOVE ASYMPTOTIC RESULT AS THE GAP BETWEEN SPHERE AND PLANE TENDS TO ZERO

Suppose that the sphere center is now situated at $z = a(1+\varepsilon)$, where the dimensionless gap ε is small, but otherwise the problem defined by equations (1)–(3) is unchanged. Cone [2] and others [1, 3, 4] showed that for the conventional conditions $T = T_0$ and T_1 on the sphere and plane, respectively, i.e. $\delta_0 = 0 = \delta_1$, the heat flow Q is given asymptotically by the expression

$$Q \sim 2\pi(T_1-T_0)ka\ln(\varepsilon^{-1}) \quad (11)$$

in which the coefficient is identical to that in equation (10). Cone [2] observed that the outer solution ($\varepsilon = 0$) is valid as a first approximation except in the gap region, where $z/a = O(\varepsilon)$; i.e. $r/a = O(\varepsilon^{1/2})$, and hence η is $O(\varepsilon^{-1/2})$ or larger. But examination of the solution from (5) shows that, for these large values of η, the dominant contribution to Q arises from values of v up to order $\varepsilon^{1/2}$. Since the outer solution ($\varepsilon = 0$) in the conventional calculation is identical to the singular solution ($\delta_0 = 0 = \delta_1$) in the calculation involving fictitious conditions, the deductions of equations (10) and (11) from equation (9) are essentially the same.

The obvious inference from this discussion is that the introduction of normal derivative terms into equations (2) and (3) has an insignificant effect on the temperature distribution unless the dimensionless gap ε is of the same order of smaller than $(\delta_0+\delta_1)$. The asymptotic estimate (11) of the heat flux must be replaced by equation (10) as ε is reduced through values of order $(\delta_0+\delta_1)$. For $\varepsilon \gg (\delta_0+\delta_1)$, introduction of the fictitious conditions causes only minor changes to the analysis given by Cone [2], and equation (11) remains appropriate. When $\varepsilon = O(\delta_0+\delta_1)$, estimates (10) and (11) are identical and valid because the outer solution ($\varepsilon = 0 = \delta_0+\delta_1$) provides a first approximation, except in a gap region whose radius is of order $a\varepsilon^{1/2}$ or $a(\delta_0+\delta_1)^{1/2}$. At such values of ε, the heat flow estimate has already attained that obtained in equation (10) for $\varepsilon = 0$, and surely cannot become any larger as ε is reduced from $O(\delta_0+\delta_1)$ to zero. Indeed, for $\varepsilon \ll (\delta_0+\delta_1)$ there exists an inner gap region, and hence determined by ε within the gap region, and hence determined above by $(\delta_0+\delta_1)$. But, in the former case, $\delta_0/\varepsilon \gg 1$ and/or $\delta_1/\varepsilon \gg 1$ imply almost perfect thermal insulation on the sphere and/or plane, respectively, and hence assure the existence of an almost constant temperature in this inner gap—which, therefore, does not generate the terms that produced the $\ln(\varepsilon^{-1})$ contributions in Cone's analysis [2]. In this way, the singularity with respect to the gap between the sphere and plane is removed by introduction of the fictitious boundary conditions of the third kind characterized by small coefficients multiplying the normal derivatives.

4. RELEVANCE OF THE FICTITIOUS BOUNDARY CONDITION TO SURFACE ROUGHNESS

A simple model is presented here to give credence to the idea of using the fictitious boundary condition of the third kind to take account of surface roughness which, due to the limitations of machine polishing, can never be entirely eliminated.

First, however, the negligible impact on the temperature field (if the changed conditions are applied on well-separated boundaries) can be illustrated by noting that the temperature distribution $T = T_0+(T_1-T_0)x/a$, determined by $d^2T/dx^2 = 0$, $T(0) = T_0$ and $T(a) = T_1$, is such that

$$T-\delta a\frac{dT}{dx} = T_0 \quad \text{at } x = a\delta \quad (12)$$

$$T+\delta a\frac{dT}{dx} = T_1 \quad \text{at } x = a(1-\delta). \quad (13)$$

Thus, T is unchanged if the conventional conditions at the flat, well-separated boundaries are replaced by fictitious conditions at fictitious surfaces that take account of roughness on a scale of $a\delta(\delta \ll 1)$.

Next, the principal example here concerns conduction between two closely-proximate surfaces, held at distinct temperatures T_0 and T_1, that are parallel except for randomly distributed, non-overlapping, spherical bumps, with a the sphere radius and each bump bounded by a circle of radius c, as in Fig. 1. The model is enhanced by restricting the ratio a/c to the range $1 \leqslant a/c \leqslant 2$; that is, the bumps protrude at least as much as caps of angle 30°, but no more than hemispheres. The separation distance can become arbitrarily small, i.e.

$$d = 2(a-\sqrt{a^2-c^2})+c\varepsilon \quad (\varepsilon \ll 1) \quad (14)$$

and so a large contribution to the heat flux, of dimensionless order $\ln(\varepsilon^{-1})$ can occur if two proximate bumps on opposite planes are almost aligned.

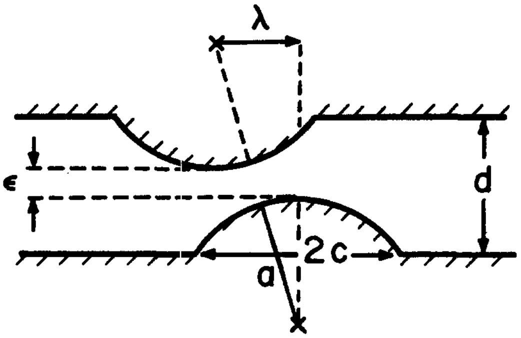

Fig. 1. Cross-sectional geometry of two closely-proximate spherical bumps on parallel surfaces spaced at a distance of d units apart.

Let $p(\lambda)$ be the probability density that the lowest point on a bump situated on the upper plane is horizontally misaligned from the nearest highest point on the lower boundary by a distance $\leqslant \lambda$. Evidently, $p(\lambda) = O(\lambda^2)$ as $\lambda \to 0$, and this property is crucial to the subsequent argument. For small enough λ, a large heat flux of the type $2\pi(T_1 - T_0)ka\ln(a/2c\varepsilon)$, described by Cone [2], occurs at each bump. If $(4b^2)^{-1}(b > c)$ is the number density of bumps on each surface, then the dominant contribution of the term that is singular in ε, to the mean heat flux per unit area, is

$$\frac{\pi(T_1 - T_0)ka}{2b^2}\int_0^{\lambda_0} \ln\left[\frac{a/2}{[(2a+c\varepsilon)^2+\lambda^2]^{1/2}-2a}\right]p'(\lambda)\,\mathrm{d}\lambda \tag{15}$$

which is not singular at $\varepsilon = 0$. Now, for a uniform array of bumps, the probability p depends on λ/b only,† so it might appear that in this case the limit value of the above expression is of the form $k(T_1 - T_0)/2bW^*(b/a)$, which cannot be correct since it takes no account of the size of the bumps and the consequent wall spacing. Geometrical considerations indicate dependence on both c/b (the fractional area covered by the bumps in $\pi c^2/4b^2$) and c/a (related to bump height by the given formula for $\lim_{\varepsilon\to 0} d/c$). In fact, the logarithmic estimate of the actual heat flux is only valid for some λ_0 of order c, with the scaling factor depending on the sharpness of the bumps. The only purpose of the discussion of the integral is to demonstrate that the mean heat flux density, which is obviously larger than $k(T_1 - T_0)/d$, possesses a finite limit value $k(T_1 - T_0)/2cW(b/c, a/c)$ as $\varepsilon \to 0$, where W satisfies the inequality

$$W(b/c, a/c) < \frac{1}{2}\lim_{\varepsilon\to 0}\frac{d}{c} = \frac{a}{c} - \left(\frac{a^2}{c^2} - 1\right)^{1/2}. \tag{16}$$

† For example, for a uniform square array of bumps, with H the Heaviside function,

$$p'(\lambda) = \frac{\lambda}{b^2}\left[\frac{\pi}{2} - 2H(\lambda - b)\cos^{-1}\left(\frac{b}{\lambda}\right)\right]$$

i.e.

$$p(\lambda) = \frac{\lambda^2}{b^2}\left[\frac{\pi}{4} - H(\lambda - b)\left\{\cos^{-1}\left(\frac{b}{\lambda}\right) - \frac{b}{\lambda}\left(1 - \frac{b^2}{\lambda^2}\right)^{1/2}\right\}\right]$$

which is such that $p(b\sqrt{2}) = 1$.

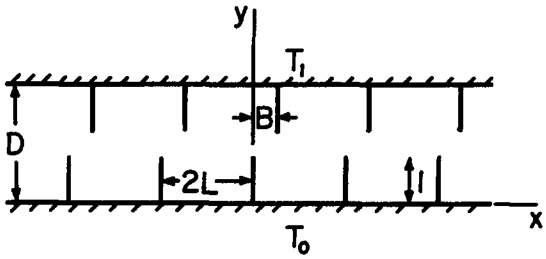

Fig. 2. Two-dimensional periodic arrays of unit length protrusions, attached to opposite walls of a channel of width D and held at different temperatures T_1 and T_0.

5. PERIODIC ARRAYS OF PROTRUSIONS IN A CHANNEL

To illustrate the ideas presented above, consider the 2-D heat flow between parallel, conducting planes that is disturbed by periodic arrays of thin, conducting protrusions that are perpendicular to the walls and not necessarily aligned, as in Fig. 2. Cartesian coordinates are chosen so that the x-axis coincides with one wall, held at temperature T_0, and the y-axis is directed towards the other wall, whose temperature is T_1. A suitable length scale is that of the protrusion, in terms of which the other lengths are, respectively: D, the distance between the walls; $2L$, the separation between neighboring barriers; and B, the misalignment of the two arrays. Thus, the temperature distribution $T = T_1(y/D) + T_0(1 - y/D)$ is disturbed by protrusions situated at $x = 2nL(-\infty < n < \infty)$, $0 \leqslant y \leqslant 1$ and $x = 2nL + B(-\infty < n < \infty)$, $D - 1 \leqslant y \leqslant D$, where $0 \leqslant B \leqslant L$, $D > 2$ and L is $O(1)$. The two sets of protrusions will be referred to respectively as 'upper' and 'lower'. The dimensionless temperature $\theta(x, y)$, defined by

$$T = T_1(y/D) + T_0(1 - y/D) + (T_1 - T_0)\theta \tag{17}$$

must satisfy the equation of steady heat flow,

$$\nabla^2\theta = 0 \tag{18}$$

and the boundary conditions

$$\theta = 0 \quad \text{at } y = 0, D \tag{19}$$

$$\theta = 1 - y/D \quad \text{or} \quad -y/D \tag{20}$$

on the upper or lower protrusions, respectively.

The temperature distribution due to an infinite array of unit heat sinks at $(2nL, y_0)$ and unit heat sources at $(2nL + B, D - y_0)$ is evidently

$$\frac{1}{4\pi}\ln\left\{\frac{\cosh[\pi(y-y_0)/L]-\cos(\pi x/L)}{\cosh[\pi(y+y_0)/L]-\cos[\pi(x-B)/L]}\right\}$$

$$=\frac{1}{4L}(|y-y_0|-|D-y-y_0|)$$

$$+\frac{1}{2\pi}\sum_{m=1}^{\infty}\frac{1}{m}\{-\mathrm{e}^{-m\pi|y-y_0|/L}\cos(m\pi x/L)$$

$$+\mathrm{e}^{-m\pi|D-y-y_0|/L}\cos[m\pi(x-B)/L]\}.$$

Consequently, an appropriate Green's function, characterized by the above sources and sinks, and satisfying equations (18) and (19), is given by

$$G(x,y;y_0)=$$

$$\frac{1}{4\pi}\ln\left\{\frac{\cosh[\pi(y-y_0)/L]-\cos(\pi x/L)}{\cosh[\pi(y+y_0)/L]-\cos[\pi(x-B)/L]}\right\}$$

$$+\frac{(y-\frac{1}{2}D)(y_0-\frac{1}{2}D)}{DL}+\frac{1}{2\pi}\sum_{m=1}^{\infty}\frac{1}{m\sinh(m\pi D/L)}$$

$$\times\langle\{\mathrm{e}^{-m\pi D/L}\cosh[m\pi(D-y-y_0)/L]$$

$$-\cosh[m\pi(y-y_0)/L]\}\cos[m\pi(x-B)/L]$$

$$+\{\cosh[m\pi(D-y-y_0)/L]$$

$$-\mathrm{e}^{-m\pi D/L}\cosh[m\pi(y-y_0)/L]\}\cos(m\pi x/L)\rangle. \quad (21)$$

The additional temperature field, θ, due to the infinite array of protrusions on either wall, can now be represented as that due to a distribution of the above periodic array of point heat singularities in the presence of the walls. Thus,

$$\theta=\int_0^1 f(y_0)G(x,y;y_0)\,\mathrm{d}y_0 \quad (22)$$

whence, from equations (21) and (22),

$$\left[\frac{\partial\theta}{\partial x}\right]_{0-}^{0+}=\begin{cases}f(y) & (0<y<1)\\ 0 & (1<y<D)\end{cases}$$

and

$$\left(\frac{\partial\theta}{\partial y}\right)_{y=0}=\int_0^1 f(y_0)\left[\frac{y_0-\frac{1}{2}D}{DL}\right.$$

$$\left.+\text{Fourier series in } x\right]\mathrm{d}y_0.$$

The total heat flux per channel length $2L$ into the lower wall and protrusions is therefore

$$k\frac{T_1-T_0}{D}\left[2L+D\int_{-L}^{L}\left(\frac{\partial\theta}{\partial y}\right)_{y=0}\mathrm{d}x+D\int_0^1 f(y)\,\mathrm{d}y\right]$$

$$=k\frac{T_1-T_0}{D}\left[2L+\int_0^1 f(y_0)(2y_0-D)\,\mathrm{d}y_0+D\int_0^1 f(y)\,\mathrm{d}y\right].$$

Thus, the heat flux Q per unit length of the channel, which is the mean heat flux density, is given by

$$Q=k\frac{T_1-T_0}{D}\left[1+\frac{1}{L}\int_0^1 f(y_0)y_0\,\mathrm{d}y_0\right]. \quad (23)$$

The function $f(y)$ is determined by solving the integral equation obtained by applying condition (20) at the protrusion situated at $x=0$, $0\leqslant y\leqslant 1$. This requires that

$$\int_0^1 f(y_0)G(0,y;y_0)\,\mathrm{d}y_0=-y/D \quad (0\leqslant y\leqslant 1). \quad (24)$$

Following the method used by Davis [9], it is noted from equation (21) that

$$G(0,y;y_0)=\frac{1}{2\pi}\left[\ln\left(\frac{|y-y_0|}{y+y_0}\right)+G^\bullet(y,y_0)\right] \quad (25)$$

where $G^\bullet$ is a regular function of y and y_0 given by

$$G^\bullet(y,y_0)=\ln\left(\frac{\sinh[\pi(y-y_0)/2L]}{y-y_0}\right)$$

$$-\ln\left(\frac{\sinh[\pi(y+y_0)/2L]}{y+y_0}\right)+\frac{2\pi yy_0}{LD}$$

$$-\tfrac{1}{2}\ln[1-2\,\mathrm{e}^{-\pi(D-y-y_0)/L}\cos(\pi B/L)+\mathrm{e}^{-2\pi(D-y-y_0)/L}]$$

$$+\sum_{m=1}^{\infty}\frac{1}{m}\left\{[\cos(m\pi B/L)+\mathrm{e}^{-m\pi D/L}]\right.$$

$$\times\frac{2\sinh(m\pi y/L)\sinh(m\pi y_0/L)}{\sinh(m\pi D/L)}$$

$$\left.-\mathrm{e}^{-m\pi(D-y-y_0)/L}\cos(m\pi B/L)\right\}. \quad (26)$$

Moreover, the expansion

$$-\ln\left(\frac{|y-y_0|}{y+y_0}\right)=4\sum_{n=1}^{\infty}\frac{1}{2n-1}T_{2n-1}(y)T_{2n-1}(y_0)$$

in terms of Chebyshev polynomials of odd order, enables the integral equation of the first kind, equation (24), to be transformed into an infinite system of linear equations of the second kind by writing

$$f(y_0)=\frac{1}{\pi}(1-y_0^2)^{-1/2}\sum_{n=1}^{\infty}(2n-1)^{1/2}f_nT_{2n-1}(y_0) \quad (27)$$

to obtain

$$f_n-\frac{4}{\pi^2}(2n-1)^{1/2}\sum_{m=1}^{\infty}(2m-1)^{1/2}f_m$$

$$\times\int_0^{\pi/2}\int_0^{\pi/2}G^\bullet(\cos\theta,\cos\chi)\cos(2n-1)\theta$$

$$\times\cos(2n-1)\chi\,\mathrm{d}\theta\,\mathrm{d}\chi=\delta_{n1} \quad (n\geqslant 1).$$

The use of finite Fourier transforms then yields the truncated system

$$f_n - \sum_{m=1}^{\infty} g_{nm} f_m = \delta_{n1} \quad (1 \leqslant n \leqslant N) \tag{28}$$

where

$$g_{nm} = \frac{1}{N^2}(2n-1)^{1/2}(2m-1)^{1/2} \times \sum_{i=0}^{N-1}{}' \sum_{j=0}^{N-1}{}' G^{\bullet}\left(\cos\frac{i\pi}{2N}\cos\frac{j\pi}{2N}\right) \times \cos\frac{(2n-1)\pi i}{2N}\cos\frac{(2m-1)\pi j}{2N}.$$

Here, the primed summations denote that a 1/2 factor is to be inserted whenever $i = 0$ and/or $j = 0$.

After substitution of equation (27) into equation (23), the mean heat flux density is now given by

$$Q = k\frac{T_1 - T_0}{D}\left[1 + \frac{f_1}{4L}\right] \tag{29}$$

and $\langle f_1 \rangle$ will denote the mean value of f_1 as B is varied over the range 0–L. Now, if $\hat{T}(y)$ denotes a temperature distribution that satisfies the fictitious boundary conditions

$$\hat{T} + \delta\frac{d\hat{T}}{dy} = T_1 \quad \text{at } y = Y_1$$

$$\hat{T} + \delta\frac{d\hat{T}}{dy} = T_0 \quad \text{at } y = Y_0 \tag{30}$$

then the corresponding heat flux density, $\hat{Q}$, is given by

$$\hat{Q} = k\frac{T_1 - T_0}{Y_1 - Y_0 + 2\delta}.$$

The values of $Y_1 - Y_0$ and δ are to be chosen by comparison of $\hat{Q}$ with Q, given by equation (29). First, the zero gap limit suggests that the impedance coefficient δ be defined by

$$\delta = \left[1 + \frac{1}{4L}\lim_{D\to 2}\langle f_1 \rangle\right]^{-1} \tag{31}$$

for given L. Then $Y_1 - Y_0$ is estimated by setting $\hat{Q} = Q$.

Equation (28) was solved numerically (with $N = 10$, 20, 40 to establish the convergence pattern) to determine values of f_1 for various values of L of order unity and with D approaching 2. The results for δ and $Y_1 - Y_0$ are displayed in Table 1, with the latter quantity being best tabulated in terms of its deviation from the 'gap' width $D-2$. The consistently small values of $[Y_1 - Y_0 - (D-2)]$ and their insensitivity to variations in L and D (except near $D = 2$) indicate that the fictitious conditions, equation (30), are to be applied almost on the lines drawn through the end points of the protrusions. This remains true for $L = 2$ (separation $= 4 \times$ length of protrusion), with the compensating factor being an increase in δ to a value where the flux plays a significant role in each boundary condition. Evidently $\delta \to 0$ as $L \to 0$, in which limit a uniform channel of width $D-2$ is obtained.

Table 1. Values of the impedance coefficient δ for various protrusion semi-separations and, for the same values of L, values of the difference, $Y_1 - Y_0 - (D-2)$, between the effective channel width for application of the fictitious boundary conditions and the distance of the nearest possible approach of the protrusions, for various values of D tending to 2

	L			
	0.5	1.0	1.5	2.0
	δ			
	0.208	0.394	0.528	0.619
D	$Y_1 - Y_0 - (D-2)$			
3.0	0.024	0.041	0.046	0.045
2.5	0.024	0.040	0.043	0.041
2.2	0.023	0.035	0.035	0.032
2.1	0.021	0.028	0.027	0.022
2.05	0.017	0.021	0.020	0.017
2.025	0.013	0.013	0.013	0.011

The results displayed in Table 1, corresponding to a wide range of values of L and D, indicate a robust applicability of the idea that a fictitious boundary condition can be applied at the line drawn through the tips of the protrusions, with the impedance coefficient increasing with the separation parameter L, as expected. These particular protrusions can be regarded as the limiting case, in two dimensions, of extreme roughness. The efficient use of impedance conditions in this case argues well for its success with other geometries.

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